— Dr. S. Chakraborty

Elementary operations on the matrix:

An elementary operation on a matrix A over a field F is an operation of the following types:

1. Interchange of two rows (or columns) of A.
2. Multiplication of a row (or column) by a non-zero scalar $c \in F$.
3. Addition of a scalar multiple of one row (or column) to another row (or column).

Remarks:

(i) When the elementary operations are applied to rows, they are said to be elementary row operations.

(ii) When the elementary operations are applied to columns, they are said to be elementary column operations.

(iii) If T be an elementary operation on A such that $T(A) = B$ and T_1 be an elementary operation on B such that $T_1(B) = C$, then $C = T_1\{T(A)\}$.

(iv) The inverse of T, denoted by T^{-1}, is defined to be an elementary operation s.t. $T^{-1}\{T(A)\} = A$.

For example, if $T = R_{ij}$, then $T^{-1} = R_{ij}$;
if $T = R_i(c)$, then $T^{-1} = R_i(c^{-1})$;
if $T = R_i + cR_j$, then $T^{-1} = R_i - cR_j$.

So, T^{-1} is an elementary operation of the same type of T.

Note: R_{ij} denotes the interchange of the ith and jth row.
cR_i or, $R_i(c)$ " multiplication of the ith row by scalar c.
$R_i + cR_j$ denotes addition of c times jth row to the ith row.
Similar notation follows for elementary column operations.

Row equivalent and Column equivalent matrices.

A matrix B over a field F is said to be row equivalent (column equivalent) to a matrix A over F if B can be obtained by successive application of a finite number of elementary row operations (column operations) on A.

For example, let us consider a matrix.

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 2 & 4 \end{pmatrix} \xrightarrow{R_{23}} \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 4 \\ 1 & 0 & 1 \end{pmatrix} \xrightarrow{R_3+2R_1} \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 4 \\ 5 & 2 & 5 \end{pmatrix} = B.$$

Here matrix B is row equivalent to matrix A.

Row-reduced matrix:

Definition → An $m \times n$ matrix A is said to be a row-reduced if

(i) the first non-zero element in each non-zero row is the leading 1; and

(ii) in each column containing the leading 1 of some row is the only non-zero element.

Examples:

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{3\times 5}, \quad B = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 3 & 0 & 2 & 1 \\ 0 & 1 & 0 & 4 & 2 \end{pmatrix}_{3\times 5}.$$

↑ Row-reduced (A)

→ NOT Row-reduced (B)

Row echelon matrix:

Definition → An $m \times n$ matrix A is said to be a row-reduced echelon matrix if

(i) A is row-reduced;

(ii) there is an integer r $(0 \le r \le m)$ s.t. the first r rows are non-zero rows and the remaining rows, if exist, are all zero rows;

(iii) if the leading 1 of the ith non-zero row occurs in the K_ith column, then $K_1 < K_2 < \cdots < K_r$.

Examples:

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

A, B → Row-echelon matrices

C → NOT Row-echelon.

1. **Theorem** → A matrix A can be made row equivalent to a row echelon matrix B by elementary row operations.

2. **Theorem** → The rank of a matrix remains invariant under ~~an~~ elementary row operations (or elementary column operations).

3. **Theorem** → If a row echelon matrix R has r non-zero rows, then rank of R = r.

4. **Theorem** → If a matrix A be row equivalent to a row echelon matrix having r non-zero rows, then rank of A = r.

Note: Two row equivalent matrices have the same rank.

Fully reduced normal form:

- **Definition** → An $m \times n$ matrix B is said to be equivalent to an $m \times n$ matrix A over the field F, if B is obtained from A by a finite number of elementary row and column operations.

- **Definition** → Fully reduced normal form of a matrix $A_{m \times n}$ is obtained by reducing A into a row-reduced echelon matrix and then into a column-reduced echelon matrix.

So, the fully reduced normal form of matrix has the following properties—

(i) No zero row (column) is followed by a non-zero row (column).

(ii) The leading 1 in each non-zero row (column) is the only non-zero element.

(iii) the leading 1 in the kth row is the leading 1 in the kth column.

Fully reduced normal form:

$$\begin{pmatrix} I_r & O_{r,n-r} \\ O_{m-r,r} & O_{m-r,n-r} \end{pmatrix}$$

Examples:

$$\left(\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)_{6\times 5} \equiv \begin{pmatrix} I_3 & O_{3,2} \\ O_{3,3} & O_{3,2} \end{pmatrix}.$$

Exercises.

① Reduce the matrix A to row-reduced form and row echelon matrix:

$$A = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & 1 & 4 & 6 \\ 3 & 0 & 7 & 9 \end{pmatrix}$$

Applying elementary row operations, we get—

$$A \xrightarrow[R_3-3R_1]{R_2-2R_1} \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \xrightarrow{R_1-2R_3} \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = R.$$

R is both row-reduced and row echelon matrix.

Ex. ⑥ Find all real λ for which the rank of the matrix A is 2.

(i) $A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 5 & 3 & \lambda \\ 1 & 1 & 6 & \lambda+1 \end{pmatrix}$

Applying elementary row operations, we obtain—

$$A \xrightarrow[R_3-R_1]{R_2-2R_1} \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & -3 & \lambda-2 \\ 0 & -1 & 3 & \lambda \end{pmatrix} \xrightarrow[R_3+R_2]{R_1-2R_2} \begin{pmatrix} 1 & 0 & 9 & 5-2\lambda \\ 0 & 1 & -3 & \lambda-2 \\ 0 & 0 & 0 & 2\lambda-2 \end{pmatrix} = R$$

If $2\lambda - 2 = 0$, i.e., $\lambda = 1$, then the rank of $A = 2$.
[Because, if $\lambda = 1$, then the row equivalent matrix R has two non-zero rows].

(ii) $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & \lambda \\ 5 & 7 & 1 & \lambda^2 \end{pmatrix}$

Applying elementary row operations, we obtain—

$$A \xrightarrow[R_3-5R_1]{R_2-R_1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & \lambda-1 \\ 0 & 2 & -4 & \lambda^2-5 \end{pmatrix} \xrightarrow[R_3-2R_2]{R_1-R_2} \begin{pmatrix} 1 & 0 & 3 & 2-\lambda \\ 0 & 1 & -2 & \lambda-1 \\ 0 & 0 & 0 & \lambda^2-2\lambda-3 \end{pmatrix} = R.$$

If $\lambda^2 - 2\lambda - 3 = 0$, i.e., $(\lambda-3)(\lambda+1) = 0$, or, $\underline{\lambda = -1, 3}$; then the rank of $A = 2$.

Ex. Find a row echelon matrix which is row equivalent to:

$$\begin{pmatrix} 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{pmatrix}$$

Let us apply elementary row operations:

$$\xrightarrow{R_{12}} \begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{pmatrix} \xrightarrow[R_4-3R_1]{R_3-2R_1} \begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & -5 & -2 & 3 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{2}R_2} \begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & -5 & -2 & 3 \end{pmatrix} \xrightarrow[\substack{R_3+2R_2 \\ R_4+5R_2}]{R_1-2R_2} \begin{pmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 3 \end{pmatrix} \xrightarrow{R_{34}} \begin{pmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{3}R_3} \begin{pmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[R_2-R_3]{R_1-2R_3} \begin{pmatrix} 1 & 3 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = R,$$ row echelon matrix

Rank of A = Rank of $R = \underline{3}$ (3 non-zero rows).

To find the fully reduced normal form:

$$R \xrightarrow[C_5+C_1]{C_2-3C_1} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{C_5+C_3} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{C_5-C_4}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{C_{23}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{C_{34}} \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$= \begin{pmatrix} I_3 & O_{3,2} \\ O_{1,3} & O_{1,2} \end{pmatrix}.$$ This is fully reduced normal form.

Syestem of Linear Equations:

- A system of m linear equations in n unknowns $x_1, x_2, \ldots, x_n$ is of the form:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \longrightarrow (1)$$

- where a_{ij}'s and b_i's are elements of a field F, called the field of scalars.
a_{ij}'s are called coefficients of the system.
a_{ij}'s and b_i's are real or complex numbers when F is the field $\mathbb{R}$ or $\mathbb{C}$.

- **The Matrix Equation:**
A matrix equation is an equation of the form $AX = b$, where A is an $m \times n$ matrix, b is a vector in R^m, and X is a vector in R^n whose coefficients a_{ij}'s are unknown.

- The matrix representation of the system (1) is:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}; \text{ i.e., } AX = b \longrightarrow (2)$$

where $A = (a_{ij})_{m\times n}$, $X = (x_1, x_2, \ldots, x_n)^t$, $b = (b_1, b_2, \ldots, b_m)^t$.

The matrix A is called the coefficient matrix,
and " " $\tilde{A} = (A, b)$ " " " augmented matrix
of the system.

Vector Equation:

Let $v_1, v_2, \cdots, v_n$ and b be vectors in $\mathbb{R}^m$.
Let us consider the vector equation:

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b. \longrightarrow (3) \quad \left[\begin{array}{l}\text{A linear Combination}\\ \text{of vectors}\end{array}\right]$$

This is equivalent to the matrix equation:

$AX = b$, where
$$A = \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & \cdots & v_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix} \end{array} \text{ and } X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

- Conversely, if A is any $m \times n$ matrix, then the matrix equation $AX = b$ is equivalent to the vector equation $x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b$, where $v_1, v_2, \cdots, v_n$ are the columns of A, and $x_1, x_2, \cdots, x_n$ are the elements of X.

Therefore, a system of linear equations, a matrix equation, and a vector equation are all equivalent.

Four equivalent Ways of Writing a Linear System:

Let us take an example:

1. As a system of linear equations:

$$\begin{cases} 2x_1 + 3x_2 - 2x_3 = 7 \\ x_1 - x_2 - 3x_3 = 5 \end{cases}$$

2. As an augmented matrix:

$$\left(\begin{array}{ccc|c} 2 & 3 & -2 & 7 \\ 1 & -1 & -3 & 5 \end{array}\right).$$

3. As a vector equation: $x_1 v_1 + x_2 v_2 + x_3 v_3 = b$;

$$x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}.$$

4. As a matrix equation: $AX = b$;

$$\begin{pmatrix} 2 & 3 & -2 \\ 1 & -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}.$$

- In particular, all four have the same solution set.

Consistent and inconsistent :

A system of equations (1) is said to be consistent if it has a solution.
Otherwise, it is said to be inconsistent.

Solution of the system (1) :

An ordered set $(c_1, c_2, \cdots, c_n)$ where $c_i \in F$, is said to be a solution of the system (1) if each equation of the system is satisfied by

$x_1 = c_1,\ x_2 = c_2,\ \cdots,\ x_n = c_n$.

$\therefore$ A solution of the system (1) is an n-tuple vector in R^n.

Examples :

(1) $\left.\begin{aligned} x + 2y &= 7 \\ 3x - y &= 5 \end{aligned}\right\}$ It has one and only one solution $(17/7, 16/7)$. System is consistent.

(2) $\left.\begin{aligned} 3x + 2y &= 4 \\ 6x + 4y &= 8 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} 3x+2y&=4 \\ 3x+2y&=4 \end{aligned}\right\}$ It has infinitely many solutions: $(4/3 - 2/3 k, k) = (4/3, 0) + k(-2/3, 1)$, for all $k \in \mathbb{R}$.

System is consistent.

(3) $\left.\begin{aligned} 4x + 3y &= 5 \\ 8x + 6y &= 8 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} 4x+3y&=5 \\ 4x+3y&=4 \end{aligned}\right\}$ It has no solution. System is inconsistent.

Remark:

1. The system $AX = b$ is said to be a homogeneous system if $b = 0$; otherwise, a non-homogeneous system.

2. Two systems $AX = b$ and $CX = d$ are said to be equivalent systems if the augmented matrices (A, b) and (C, d) be row equivalent.

3. For two equivalent systems $AX = b$ and $CX = d$, if α be a solution of $AX = b$, then α is also a solution of $CX = d$.

Homogeneous System : $AX = 0 \longrightarrow (1)$

PROPERTIES :–

(1) The homogeneous system is necessarily a consistent one; since $(0, 0, \cdots, 0)$ in R^n is always its solution, called trivial solution.

(2) Theorem : The solutions of $AX = 0$ in n unknowns form a subspace of $V_n(F)$, where A is an $m \times n$ matrix over a field F.

Proof : Let S be the set of all solutions of the system (1). Each solution is an n-tuple vector in $V_n(F)$.

Case 1. When the trivial (zero) solution is the only solution of (1), then $S = \{\theta\}$, which is a subspace of $V_n(F)$.

Case 2. When there is a non-zero solution, say,

$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in S$. Let $c \in F$.

$\therefore \quad a_{i1}\alpha_1 + a_{i2}\alpha_2 + \cdots + a_{in}\alpha_n = 0$; for $i = 1, 2, \cdots, m$.

$\Rightarrow a_{i1}(c\alpha_1) + a_{i2}(c\alpha_2) + \cdots + a_{in}(c\alpha_n) = 0$; for $i = 1, 2, \cdots, m$.

$\Rightarrow (c\alpha_1, c\alpha_2, \cdots, c\alpha_n) \in S \Rightarrow c\cdot\alpha \in S$

$\therefore \alpha \in S,\ c \in F \Rightarrow c\cdot\alpha \in S \longrightarrow$ (i)

Let $\beta = (\beta_1, \beta_2, \cdots, \beta_n) \in S$.

$\therefore \quad a_{i1}\beta_1 + a_{i2}\beta_2 + \cdots + a_{in}\beta_n = 0$; for $i = 1, 2, \cdots, m$

$\therefore a_{i1}(\alpha_1+\beta_1) + a_{i2}(\alpha_2+\beta_2) + \cdots + a_{in}(\alpha_n+\beta_n) = 0$; $i = 1, 2, \cdots, m$.

$\left[\text{Since } \sum_{j=1}^{n} a_{ij}\alpha_j = 0 = \sum_{j=1}^{n} a_{ij}\beta_j \text{ ; for } i = 1, 2, \cdots, m.\right]$

$\Rightarrow (\alpha_1+\beta_1, \alpha_2+\beta_2, \cdots, \alpha_n+\beta_n) \in S \Rightarrow \alpha+\beta \in S.$

$\therefore \quad \alpha \in S,\ \beta \in S \Rightarrow \alpha+\beta \in S \longrightarrow$ (ii)

From (i) & (ii) it follows that S is a subspace of $V_n(F)$.

NOTE: S is denoted by $X(A)$, called the solution space.

rank of A + rank of $X(A) = n$.

If A have r independent column vectors, then rank of $A = r \Rightarrow$ rank of $X(A) = n - r$.

③ If the number of equations be less than the number of unknowns (i.e., $m < n$) in the system $AX = 0$, then the system admits a non-zero solution (infact, infinitely many solutions).

Because, since $m < n$, rank of $A < n$.

$\therefore$ rank of A + rank of $X(A) = n$

$\Rightarrow$ rank of $X(A) > 0 \Rightarrow \exists$ a non-zero solution.

④ The homogeneous system $AX = 0$ containing n equations in n unknowns has a non-zero solution iff rank of $A < n$ (= the number of unknowns). The number of solutions, in fact, is infinite.

NON-HOMOGENEOUS System: $AX = b$.

The system may not have a solution.

Necessary and sufficient condition that the system $AX = b$ to be consistent is rank of A = rank of $\tilde{A}$.

Note: The set of solutions of a consistent non-homogeneous system $AX = b$ does not form a subspace of $V_n(F)$, because $(0, 0, \cdots, 0)$ is not a solution.

Theorem: If the non-homogeneous system $AX = b$ possesses a solution X_0 (particular solution), then all solutions (general solution) of the system are obtained by adding X_0 to the general solution of its associated homogeneous system $AX = 0$.

Corollary: If $AX = b$ is consistent, then the system possesses only one solution or infinitely many solutions according as the associated $AX = 0$ possesses only the zero solution or infinitely many solutions.

Existence and Number of Solutions of the system $AX = b$; where $A = (a_{ij})_{m,n}$.

Case 1: $m = n$.

The system is consistent iff $r(A) = r(\tilde{A})$.

(i) If $r(A) = r(\tilde{A}) = n$, then A is non-singular $\Rightarrow A^{-1}$ exists and $X = A^{-1}b$ is a unique solution. The system $AX=0$ has the trivial solution.

(ii) If $r(A) = r(\tilde{A}) < n$, then the associated homogeneous system $AX = 0$ has infinitely many solutions.

$\therefore$ the system $AX = b$ possesses infinitely many solutions.

Case 2: $m < n$.

The system is consistent iff $r(A) = r(\tilde{A}) \leq m$
i.e., iff $r(A) = r(\tilde{A}) < n$.

$\Rightarrow$ The system $AX = b$ has infinitely many solutions according the system $AX = 0$ has.

Case 3: $m > n$.

The system is consistent iff $r(A) = r(\tilde{A}) \leq n$.

(i) When $r(A) = r(\tilde{A}) = n$,
Now $r(A) + r(X(A)) = n$, where $X(A)$ is the solution space of the system $AX = 0$.
$\Rightarrow r(X(A)) = n - r(A) = n - n = 0$.
$\therefore$ The homogeneous system $AX = 0$ possesses only the zero (trivial) solution, and hence the system $AX = b$ possesses only one solution.

(ii) When $r(A) = r(\tilde{A}) < n$, then $AX = b$ has infinitely many solutions according as the system $AX = 0$ has.

Exercises.

3.(i) Solve the equations: $\left.\begin{aligned} x+y+3z&=0 \\ 2x+y+z&=0 \\ 3x+2y+4z&=0 \end{aligned}\right\}$

This is a homogeneous system $AX=0$, where $A=\begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 1 \\ 3 & 2 & 4 \end{pmatrix}$. Let us apply elementary row operations on A to reduce it to a row reduced echelon matrix.

$$A \xrightarrow[R_3-3R_1]{R_2-2R_1} \begin{pmatrix} 1 & 1 & 3 \\ 0 & -1 & -5 \\ 0 & -1 & -5 \end{pmatrix} \xrightarrow{(-1)R_2} \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 5 \\ 0 & -1 & -5 \end{pmatrix} \xrightarrow[R_3+R_2]{R_1-R_2} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}.$$

The given system is equivalent to

$\left.\begin{aligned} x-2z&=0 \\ y+5z&=0 \end{aligned}\right\}$ Let us choose $z=c\in\mathbb{R}$, arbitrary. Then $x=2c$, $y=-5c$.

$\therefore$ The solutions are given by $(2c,-5c,c)=c(2,-5,1)$.

The solutions form a vector space generated by the vector $(2,-5,1)$ which is linearly independent [$\because \{(2,-5,1)\}$ has a non-zero, only one vector, it is l.i.d.]

$\therefore\ r(X(A))=1$, $r(A)=r(\tilde{A})=2 < n(=3)$. [$\because$ Row echelon matrix has only two non-zero l.i.d. row vectors]

And $r(A)+r(X(A))=2+1=3$ ($=n$, no. of unknowns).

$\therefore$ The given system has infinitely many solutions: $\underline{c(2,-5,1)}$, where c is an arbitrary real no.

(ii) Solve: $\left.\begin{aligned} x+y-z-w&=0 \\ x-y+z-w&=0 \end{aligned}\right\}$ This is a homogeneous system $AX=0$, where the coefficient matrix is given by

$$A=\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \xrightarrow{R_2-R_1} \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & -2 & 2 & 0 \end{pmatrix} \xrightarrow{-\frac{1}{2}R_2} \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

$\xrightarrow{R_1-R_2} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$. The given system is equivalent to:

$\left.\begin{aligned} x-w&=0 \\ y-z&=0 \end{aligned}\right\}$ Let us choose $z=c$, $w=d$ where $c,d\in\mathbb{R}$, are arbitrary.

$r(A)=r(\tilde{A})=2 < n(=4) \Rightarrow$ system has infinitely many solutions.

Solutions: $(d,c,c,d)=c(0,1,1,0)+d(1,0,0,1)$. $r(X(A))=2$.